Appendix

Proof of Proposition 1

For a customer who can afford neither product at listed price ($v_i < p_i$ for $i = 1, 2$), NYOP is the only channel in which she might get some product, so she will always attend NYOP. Her expected payoff is $V_{DC}^B(\mathbf{v}) = \max_b G(b)(\alpha_i v_j + \alpha_i v_j - b)$ The FOC of the inner function is $g(b)(\alpha_1 v_1 + \alpha_2 v_2 - b) - G(b)$. Under DRHR assumption, there exists a unique $b^* \in [0, p]$ that satisfies

$$\alpha_1 v_1 + \alpha_2 v_2 = b^* + G(b^*)/g(b^*). \tag{A1}$$

Also, as $G(b)/g(b)$ increases in b, the optimal bid increases with the expected valuation of the opaque product $\alpha_1 v_1 + \alpha_2 v_2$.

For a customer with an external choice ($v_1 \geq p_1$ and/or $v_2 \geq p_2$), the optimal bid b^* maximizes $G(b)(\alpha_1 v_1 + \alpha_2 v_2 - b) + \bar{G}(b) \max_{k=0,1,2,} V_{DC}^k(\mathbf{v})$. Without loss of generality, assume that $v_1 - p_1 \geq v_2 - p_2$, and denote $|v_i - v_j + p_j - p_i| = v_1 - v_2 + p_2 - p_1 > 0$ the *degree of differentiation*. Followed by a similar analysis as above, it is the unique value that satisfies

$$\alpha_2(v_2 - v_1) + p_1 = b^* + G(b^*)/g(b^*). \tag{A2}$$

Due to DRHR assumption, $b^* \geq 0$ if and only if $\alpha_2(v_2 - v_1) + p_1 \geq 0$. The bid increases with $v_2 - v_1$, hence decreasing in $v_1 - v_2 + p_2 - p_1$, the degree of differentiation.

The customer will choose to bid b^* at the NYOP channel first, if and only if it yields higher expected payoff than buying from direct channel 1, i.e., $V_{DC}^B(\mathbf{v}) \geq V_{DC}^1(\mathbf{v})$, which requires that $G(b^*)(\alpha_1 v_1 + \alpha_2 v_2 - b^*) + \bar{G}(b^*)(v_1 - p_1) > v_1 - p_1$, or equivalently,

$$b^* < (\alpha_1 v_1 + \alpha_2 v_2) - (v_1 - p_1) = \alpha_2(v_2 - v_1) + p_1. \tag{A3}$$

This apparently holds when $\alpha_2(v_2 - v_1) + p_1 \geq 0$. Therefore, the customer will NYOP first if and only if $\alpha_2(v_2 - v_1) + p_1 > 0$, which is equivalent to requiring that the degree of differentiation $v_1 - v_2 + p_2 - p_1 \leq p_1/\alpha_2 + p_2 - p_1 = \dfrac{\alpha_1 p_1 + \alpha_2 p_2}{\alpha_2}$. □

Proof of Theorem 1

Suppose only seller 1 is in stock, the product at the intermediary firm ceases to be an opaque product and becomes a regular one. It is obviously beneficial for the customers to NYOP in the first place, and if rejected, reconsider purchasing from the direct channel. Therefore, the customer first has to determine her bid, b, that maximizes her expected payoff:

$$G(b)(v_j - b) + \bar{G}(b) \max\{v_1 - p_1, 0\} = \begin{cases} G(b)(p - b) + (v_1 - p_1), & \text{if } v_1 \geq p_1 \\ G(b)(v_1 - b), & \text{if } v_1 < p_1. \end{cases}$$

The FOC is given by

$$g(b)\left[\min\{v_1,p_1\}-b\right]-G(b)=g(b)\left[\min\{v_1,p_1\}-b-\frac{G(b)}{g(b)}\right].$$

As $G(b)/g(b)$ increases in b, there exists a unique $b^*\in[0,\min\{v_1,p_1\}]$ such that $\min\{v_1,p_1\}-b^*-\frac{G(b^*)}{g(b^*)}=0$. It is easy to verify that the extreme solutions ($b=0$ and $b=\min\{v,p\}$) are not optimal; therefore, the expected payoff is maximized at FOC=0, i.e, when $b=b^*$. This raises the following lemma:

Lemma A1. *When only seller 1 is in stock with direct channel price p_1, a customer with valuation v_1 will place a bid b^* with the NYOP channel in the first place, which satisfies*

$$b^*+\frac{G(b^*)}{g(b^*)}=\min\{v_1,p_1\} \tag{A4}$$

By (A4), the optimal bid satisfies $b^*+\frac{G(b^*)}{g(b^*)}=\min\{v_1,p_1\}\le p_1$. Consider r_0 which satisfies

$$r_0+\frac{G(r_0)}{g(r_0)}=p_1. \tag{A5}$$

Then, obviously $b^*<r_0$, i.e., r_0 is beyond the bid of any customer. Therefore if $r\ge r_0$, a customer will be always rejected by the NYOP channel and if $v_1\ge p_1$, she will obtain the product through the direct channel. On the other hand, if $r<r_0$, a customers with $v_1\ge r+\frac{G(r)}{g(r)}$ will bid over r and be accepted by the NYOP intermediary. If one is rejected for bidding below r, i.e., $v_1<r+\frac{G(r)}{g(r)}$, she cannot afford the posted price neither since $v_1<r+\frac{G(r)}{g(r)}<r_0+\frac{G(r_0)}{g(r_0)}=p$. This proves the following lemma:

Lemma A2. *At period t, suppose r_0 is defined by* (A5) *with $p_1=p_{1t}$. The sales will only be realized through the direct channel if $r_t\ge r_0$, or the NYOP channel otherwise.*

Now consider the seller's optimal decision. With posted price p and reservation price r for a particular period, seller 1's spot revenue π_1 is given by

$$\pi_1(p,r)=\begin{cases}\bar F_1(p)p, & \text{if } r>r_0\\ \bar F_1\left(r+\frac{G(r)}{g(r)}\right)r & \text{if } r\le r_0,\end{cases}$$

where r_0 is defined in (A5).

At $t=1$, seller 1 will seek a pair (p,r) that maximizes $\pi_1(p,r)$. Denote p^* the optimal posted price if the seller determines to use direct channel, i.e., $r\ge r_0$. If the seller chooses to use the NYOP channel ($r<r_0$) only, the objective becomes

$$\max_{r<r_0}\bar F_1\left(r+\frac{G(r)}{g(r)}\right)r=\max_{r<r_0}\frac{r}{r+G(r)/g(r)}\bar F_1\left(r+\frac{G(r)}{g(r)}\right)\left(r+\frac{G(r)}{g(r)}\right)$$

$$
\begin{aligned}
&\leq \max_{r<r_0} \frac{r}{r+G(r)/g(r)} F_1(p^*)p^* \\
&< F_1(p^*)p^*.
\end{aligned}
$$

Therefore, in the last time epoch it is always optimal not to let customers purchase from the NYOP channel.

At $t > 1$, denote by $\Pi_1(x,t)$ the optimal expected profit that seller 1 will receive if inventory level is x. Then,

$$
\Pi_1(x,t) = \max_{p\geq r\geq 0} \begin{cases} \bar{F}_1(p)\,[\,p+\Pi_1(x-1,t-1)\,]+F_1(p)\Pi_1(x,t-1), & \text{if } r>r_0 \\ \bar{F}_1\left(r+\dfrac{G(r)}{g(r)}\right)[\,r+\Pi_1(x-1,t-1)\,]+F_1\left(r+\dfrac{G(r)}{g(r)}\right)\Pi_1(x,t-1) & \text{if } r\leq r_0, \end{cases}
$$

Let r^* be the optimal reservation price if the seller decides to go with NYOP selling, i.e.,

$$
r^* = \arg\max_r \bar{F}_1\left(r+\frac{G(r)}{g(r)}\right)[\,r+\Pi_1(x-1,t-1)\,]+F_1\left(r+\frac{G(r)}{g(r)}\right)\Pi_1(x,t-1).
$$

Consider $p^* = r^* + \dfrac{G(r^*)}{g(r^*)} - \epsilon < r^* + \dfrac{G(r^*)}{g(r^*)}$, where $\epsilon \to 0$. The following verifies that selling through direct channel with posted price $p_t = p^*$ yields a higher expected profit then that can be achieved via NYOP selling only:

$$
\begin{aligned}
&\bar{F}_1(p^*)\,[\,p^*+\Pi_1(x-1,t-1)\,]+F_1(p^*)\Pi_1(x,t-1) \\
=\ &\bar{F}_1\left(r^*+\frac{G(r^*)}{g(r^*)}\right)[r^*+\Pi_1(x-1,t-1)]+F_1\left(r^*+\frac{G(r^*)}{g(r^*)}\right)\Pi_1(x,t-1)+\bar{F}_1\left(r^*+\frac{G(r^*)}{g(r^*)}\right)\frac{G(r^*)}{g(r^*)} \\
&+\left[\bar{F}_1\left(r^*+\frac{G(r^*)}{g(r^*)}-\epsilon\right)-\bar{F}_1\left(r^*+\frac{G(r^*)}{g(r^*)}\right)\right]\left[r^*+\frac{G(r^*)}{g(r^*)}+\Pi_1(x-1,t-1)-\Pi_1(x,t-1)\right] \\
&-\epsilon\bar{F}_1\left(r^*+\frac{G(r^*)}{g(r^*)}-\epsilon\right).
\end{aligned}
$$

When ϵ is small enough, we have

$$
\begin{aligned}
&\left[\bar{F}_1\left(r^*+\frac{G(r^*)}{g(r^*)}-\epsilon\right)-\bar{F}_1\left(r^*+\frac{G(r^*)}{g(r^*)}\right)\right]\left[r^*+\frac{G(r^*)}{g(r^*)}+\Pi_1(x-1,t-1)-\Pi_1(x,t-1)\right] \\
&-\epsilon\bar{F}_1\left(r^*+\frac{G(r^*)}{g(r^*)}-\epsilon\right)\to 0.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
&\bar{F}_1(p^*)\,[\,p^*+\Pi_1(x-1,t-1)\,]+F_1(p^*)\Pi_1(x,t-1) \\
&\geq \bar{F}_1\left(r^*+\frac{G(r^*)}{g(r^*)}\right)[r^*+\Pi_1(x-1,t-1)]+F_1\left(r^*+\frac{G(r^*)}{g(r^*)}\right)\Pi_1(x,t-1).
\end{aligned}
$$

Therefore, it is optimal for seller 1 to use direct channel only.

We characterize the optimal price and expected profit as follows:

For $t=1$, $p_{11}^*=\arg\max_p \bar{F}_1(p)p=\arg\max_p(1-p)p=1/2$, $\Pi_1(x,1)=\bar{F}_1(p_{11}^*)p_{11}^*=1/4$.

For $t>1$,

$$\begin{aligned}
p_{1t}^*(x) &= \arg\max_p \bar{F}_1(p)\,[p+\Pi_1(x-1,t-1)]+F_1(p)\Pi_1(x,t-1)\\
&= \arg\max_p [1-p]\,[p+\Pi_1(x-1,t-1)]+p\Pi_1(x,t-1)\\
&= \arg\max_p -p^2+p[\,1+\Pi_1(x,t-1)-\Pi(x-1,t-1)\,]+\Pi_1(x-1,t-1)\\
&= \frac{1+\Pi_1(x,t-1)-\Pi_1(x-1,t-1)}{2}
\end{aligned}$$

and

$$\begin{aligned}
\Pi_1^*(x,t) &= \bar{F}_1(p_{1t}^*)\,[p_{1t}^*+\Pi_1(x-1,t-1)]+F_1(p_{1t}^*)\Pi_1(x,t-1)\\
&= \Pi_1(x-1,t-1)+\left(\frac{1+\Pi_1(x,t-1)-\Pi_1(x-1,t-1)}{2}\right)^2
\end{aligned}$$

$\square$

Proof of Theorem 2:

(i) Under the uniform valuation assumption, the final purchasing realization (1) is

$$\mathbf{H}_{SDC1}(\mathbf{v})=\begin{cases}(0,0) & \text{if } \max\{v_1-p_1,v_2-p_2\}<0 \text{ and } 2r>v_1;\\ (1,p_1) & \text{if } v_1-p_1\geq\max\{v_2-p_2,0\} \text{ and } p_1-2r<0;\\ (2,p_2) & \text{if } v_2-p_2\geq\max\{v_1-p_1,0\} \text{ and } p_2-2r<v_2-v_1;\\ (O,b^*) & \text{otherwise.}\end{cases} \tag{A6}$$

Specifically, if $2r\geq p_1$,

$$\mathbf{H}_{SDC1}(\mathbf{v})=\begin{cases}(0,0) & \text{if } \max\{v_1-p_1,v_2-p_2\}<0;\\ (1,p_1) & \text{if } v_1-p_1\geq\max\{v_2-p_2,0\};\\ (2,p_2) & \text{if } v_2-p_2\geq\max\{v_1-p_1,0\}\end{cases} \tag{A7}$$

otherwise if $2r<p_1$,

$$\mathbf{H}_{SDC1}(\mathbf{v})=\begin{cases}(0,0) & \text{if } \max\{v_1-p_1,v_2-p_2\}<0 \text{ and } 2r>v_1;\\ (2,p_2) & \text{if } v_2-p_2\geq\max\{v_1-p_1,0\} \text{ and } p_2-2r<v_2-v_1;\\ (O,b^*) & \text{otherwise.}\end{cases} \tag{A8}$$

It can be verified that for every $(p_1,r_1)=(1,r)$, the expected profit Π_1 could be enhanced by setting $(p_1,r_1)=(2r,1)$. Therefore it is optimal for seller 1 to sell through direct channel only.

(ii) We only need to prove the existence of equilibrium on (p_{1t},p_{2t}) when seller 1 choose not to sell through NYOP (i.e., $r_{1t}^*=1$, $\Pi_{1t}^*=\Pi_{1t}^D$) in period t. In particular, when $r_{1t}=r_{2t}=1$,

$$\Pi_1^D(\mathbf{x},t)=[\,p_{1t}+\Pi_1(x_1-1,x_2,t-1)\,]\,\Omega_1(\mathbf{r}_t,\mathbf{p}_t)+\Pi_1(x_1,x_2-1,t-1)\Omega_2(\mathbf{r}_t,\mathbf{p}_t)+\Pi_1(x_1,x_2,t-1)\Omega_0(\mathbf{r}_t,\mathbf{p}_t)$$

$$\Pi_2(\mathbf{x},t) = \Pi_2(x_1-1,x_2,t-1)\Omega_1(\mathbf{r}_t,\mathbf{p}_t)+[\,p_{2t}+\Pi_2(x_1,x_2-1,t-1)\,]\,\Omega_2(\mathbf{r}_t,\mathbf{p}_t)+\Pi_2(x_1,x_2,t-1)\Omega_0(\mathbf{r}_t,\mathbf{p}_t)$$

We need to prove that Π_1^D and Π_2 are unimodals in p_{1t} and p_{2t} respectively. Note that for $i=1,2$,

$$\frac{\partial\Omega_1(\mathbf{r}_t,\mathbf{p}_t)}{\partial p_i}+\frac{\partial\Omega_2(\mathbf{r}_t,\mathbf{p}_t)}{\partial p_i}+\frac{\partial\Omega_0(\mathbf{r}_t,\mathbf{p}_t)}{\partial p_i}=0.$$

Thus,

$$\begin{aligned}\frac{\partial\Pi_1^D(\mathbf{x},t)}{\partial p_{1t}} &= \Omega_1(\mathbf{r}_t,\mathbf{p}_t)\\ &\quad -\left[\widetilde{\Pi}_1(x_1,x_2,t-1)-p_{1t}\right]\frac{\partial\Omega_1(\mathbf{r}_t,\mathbf{p}_t)}{\partial p_{1t}}-[\Pi_1(x_1,x_2-1,t-1)-\Pi_1(x_1,x_2,t-1)]\,\frac{\partial\Omega_0(\mathbf{r}_t,\mathbf{p}_t)}{\partial p_{1t}}\end{aligned}$$

$$\begin{aligned}\frac{\partial\Pi_2(\mathbf{x},t)}{\partial p_{2t}} &= \Omega_2(\mathbf{r}_t,\mathbf{p}_t)\\ &\quad -\left[\widetilde{\Pi}_2(x_1,x_2,t-1)-p_{2t}\right]\frac{\partial\Omega_2(\mathbf{r}_t,\mathbf{p}_t)}{\partial p_{2t}}-[\Pi_2(x_1-1,x_2,t-1)-\Pi_2(x_1,x_2,t-1)]\,\frac{\partial\Omega_0(\mathbf{r}_t,\mathbf{p}_t)}{\partial p_{2t}}\end{aligned}$$

By (A6) for $r=1$, it can be verified that if $p_i\le p_j$,

$$\Omega_i(\mathbf{r},\mathbf{p})=1-\frac{2p_i+(1-p_j)^2}{2},\ \ \Omega_j(\mathbf{r},\mathbf{p})=\frac{(2p_i+1-p_j)(1-p_j)}{2},\ \ \Omega_0(\mathbf{r},\mathbf{p})=p_1p_2,\ \ \Omega_O(\mathbf{r},\mathbf{p})=0$$

and

$$\begin{aligned}&\frac{\partial\Omega_i(\mathbf{r}_t,\mathbf{p}_t)}{\partial p_{it}}=-1,\quad \frac{\partial\Omega_j(\mathbf{r}_t,\mathbf{p}_t)}{\partial p_{it}}=1-p_j,\quad \frac{\partial\Omega_0(\mathbf{r}_t,\mathbf{p}_t)}{\partial p_{it}}=p_j;\\ &\frac{\partial\Omega_i(\mathbf{r}_t,\mathbf{p}_t)}{\partial p_{jt}}=1-p_j,\quad \frac{\partial\Omega_j(\mathbf{r}_t,\mathbf{p}_t)}{\partial p_{jt}}=p_j-1,\quad \frac{\partial\Omega_0(\mathbf{r}_t,\mathbf{p}_t)}{\partial p_{jt}}=p_i;\\ &\frac{\partial^2\Omega_i(\mathbf{r}_t,\mathbf{p}_t)}{\partial p_{it}^2}=0,\quad \frac{\partial^2\Omega_j(\mathbf{r}_t,\mathbf{p}_t)}{\partial p_{it}^2}=0,\quad \frac{\partial^2\Omega_0(\mathbf{r}_t,\mathbf{p}_t)}{\partial p_{it}^2}=0;\\ &\frac{\partial^2\Omega_i(\mathbf{r}_t,\mathbf{p}_t)}{\partial p_{jt}^2}=-1,\quad \frac{\partial^2\Omega_j(\mathbf{r}_t,\mathbf{p}_t)}{\partial p_{jt}^2}=1,\quad \frac{\partial^2\Omega_0(\mathbf{r}_t,\mathbf{p}_t)}{\partial p_{jt}^2}=0\end{aligned}$$

Without loss of generality, assume that $i=1$ and $j=2$. Then

$$\begin{aligned}\frac{\partial\Pi_1^D(\mathbf{x},t)}{\partial p_{1t}} &= 1-p_{1t}-\frac{(1-p_{2t})^2}{2}\\ &\quad +\left[\widetilde{\Pi}_1(x_1,x_2,t-1)-p_{1t}\right]-[\Pi_1(x_1,x_2-1,t-1)-\Pi_1(x_1,x_2,t-1)]\,p_{2t}\\ \frac{\partial\Pi_2(\mathbf{x},t)}{\partial p_{2t}} &= \frac{(2p_{1t}+1-p_{2t})(1-p_{2t})}{2}\\ &\quad +\left[\widetilde{\Pi}_2(x_1,x_2,t-1)-p_{2t}\right](1-p_{2t})-[\Pi_2(x_1-1,x_2,t-1)-\Pi_2(x_1,x_2,t-1)]\,p_{1t}\end{aligned}$$

Apparently, Π_i^D is unimodal as $\dfrac{\partial\Pi_1^D(\mathbf{x},t)}{\partial p_{1t}}$ can take the value 0 at at most one p_{1t}. For Π_2, the FOC can be written as

$$\frac{\partial\Pi_2(\mathbf{x},t)}{\partial p_{2t}}=\frac{3}{2}(1-p_{2t})^2+\left[\widetilde{\Pi}_2(x_1,x_2,t-1)+\frac{p_{1t}}{2}-1\right](1-p_{2t})-[\Pi_2(x_1-1,x_2,t-1)-\Pi_2(x_1,x_2,t-1)]\,p_{1t}$$

Note that $\Pi_2(x_1-1,x_2,t-1)-\Pi_2(x_1,x_2,t-1)\geq 0$, therefore, one root (if any) of the FOC will satisfy $p_{2t}>1$. Thus, Π_2 as a function of p_{2t} is unimodal on $[0,1]$. These prove the existence of pure strategy NE. □

Proof of Proposition 2: For stage 2 at time t, the seller with lower reservation price r_i will be chosen by the NYOP intermediary as the opaque product provider. Consider seller 1, suppose his rival seller 2 proposes a reservation price $r_{2t}\geq\widetilde{\Pi}_1(\mathbf{x},t-1)$. Then, seller 1 can practically take three sets of actions:

1. *give up* the opaque provider-ship by setting $r_{1t}=1$. In this case, seller 1's expected revenue can be expressed by

$$\begin{aligned}\Pi_1^{>r}(\mathbf{p}|\mathbf{x},t) &= [\,p_1+\Pi_1(x_1-1,x_2,t-1)\,]\,\Omega_1(\mathbf{r}_t,\mathbf{p}_t)+\Pi_1(x_1,x_2-1,t-1)\Omega_O(\mathbf{r}_t,\mathbf{p}_t)\\ &\quad+\Pi_1(x_1,x_2-1,t-1)\Omega_2(\mathbf{r}_t,\mathbf{p}_t)+\Pi_1(x_1,x_2,t-1)\Omega_0(\mathbf{r}_t,\mathbf{p}_t);\end{aligned}$$

2. *match* the reservation price by setting $r_{1t}=r_{2t}$. In this case, each seller is the opaque provider with equal probability.

$$\begin{aligned}&\Pi_1^{=r}(\mathbf{p}|\mathbf{x},t)=[\,p_1+\Pi_1(x_1-1,x_2,t-1)\,]\,\Omega_1(\mathbf{r}_t,\mathbf{p}_t)\\ &+\left\{\frac{1}{2}\,[\,r+\Pi_1(x_1-1,x_2,t-1)\,]+\frac{1}{2}\Pi_1(x_1,x_2-1,t-1)\right\}\Omega_O(\mathbf{r}_t,\mathbf{p}_t)\\ &+\Pi_1(x_1,x_2-1,t-1)\Omega_2(\mathbf{r}_t,\mathbf{p}_t)+\Pi_1(x_1,x_2,t-1)\Omega_0(\mathbf{r}_t,\mathbf{p}_t);\end{aligned}$$

3. *become* the opaque provider by agreeing to a lower reservation price $r_{1t}=r_{2t}-\epsilon$. As $\epsilon\to 0$, the expected revenue is given by

$$\begin{aligned}\Pi_1^{<r}(\mathbf{p}|\mathbf{x},t) &= [\,p_1+\Pi_1(x_1-1,x_2,t-1)\,]\,\Omega_1(\mathbf{r}_t,\mathbf{p}_t)+[\,r+\Pi_1(x_1-1,x_2,t-1)\,]\,\Omega_O(\mathbf{r}_t,\mathbf{p}_t)\\ &\quad+\Pi_1(x_1,x_2-1,t-1)\Omega_2(\mathbf{r}_t,\mathbf{p}_t)+\Pi_1(x_1,x_2,t-1)\Omega_0(\mathbf{r}_t,\mathbf{p}_t).\end{aligned}$$

Comparing the three strategies, it is not hard to verify that it is optimal for seller i to *become* opaque provider whenever $r_{2t}>\widetilde{\Pi}_1(\mathbf{x},t-1)$, *match* when $r_{2t}=\widetilde{\Pi}_1(\mathbf{x},t-1)$, and *give up* when $r_{2t}<\widetilde{\Pi}_1(\mathbf{x},t-1)$.

Without loss of generality, assume that $\widetilde{\Pi}_1(\mathbf{x},t-1)\leq\widetilde{\Pi}_2(\mathbf{x},t-1)$. Then, seller 2 will compete with seller 1 for the opaque provider-ship until $r_{1t}=r_{2t}=\widetilde{\Pi}_2(\mathbf{x},t-1)$. And seller 1 can gain the full provider-ship by lowering down his reservation price to $\widetilde{\Pi}_2(\mathbf{x},t-1)-\epsilon$ where $\epsilon\to 0$. Therefore, the minimum reservation price is $r^*(\mathbf{x},t)=\widetilde{\Pi}_2(\mathbf{x},t-1)$. Seller 1 will be the unique opaque provider unless $\widetilde{\Pi}_1(\mathbf{x},t-1)=\widetilde{\Pi}_2(\mathbf{x},t-1)$, in which case both sellers are the opaque provider with equal probability. □

Proof of Theorem 3: We aim to prove that Π_i is unimodal in p_{it} for $i=1,2$. Without loss of generality, assume that $r_t = \widetilde{\Pi_2}(\mathbf{x}, t-1) < \widetilde{\Pi_1}(\mathbf{x}, t-1)$, i.e., seller 2 is the opaque-product provider. Then,

$$\begin{aligned}
\Pi_1(\mathbf{x},t) &= [p_{1t} + \Pi_1(x_1-1, x_2, t-1)]\,\Omega_1(\mathbf{r}_t,\mathbf{p}_t) + \Pi_1(x_1, x_2-1, t-1)\Omega_O(\mathbf{r}_t,\mathbf{p}_t) \\
&\quad + \Pi_1(x_1, x_2-1, t-1)\Omega_2(\mathbf{r}_t,\mathbf{p}_t) + \Pi_1(x_1, x_2, t-1)\Omega_0(\mathbf{r}_t,\mathbf{p}_t) \qquad \text{(A9a)} \\
\Pi_2(\mathbf{x},t) &= \Pi_2(x_1-1, x_2, t-1)\Omega_1(\mathbf{r}_t,\mathbf{p}_t) + \left[\widetilde{\Pi_2}(\mathbf{x}, t-1) + \Pi_2(x_1, x_2-1, t-1)\right]\Omega_O(\mathbf{r}_t,\mathbf{p}_t) \\
&\quad + [p_{2t} + \Pi_2(x_1, x_2-1, t-1)]\,\Omega_2(\mathbf{r}_t,\mathbf{p}_t) + \Pi_2(x_1, x_2, t-1)\Omega_0(\mathbf{r}_t,\mathbf{p}_t) \qquad \text{(A9b)}
\end{aligned}$$

Note that

$$-\frac{\partial \Omega_O(\mathbf{r}_t,\mathbf{p}_t)}{\partial p_i} = \frac{\partial \Omega_1(\mathbf{r}_t,\mathbf{p}_t)}{\partial p_i} + \frac{\partial \Omega_2(\mathbf{r}_t,\mathbf{p}_t)}{\partial p_i} + \frac{\partial \Omega_0(\mathbf{r}_t,\mathbf{p}_t)}{\partial p_i}$$

for $i=1,2$. Then,

$$\begin{aligned}
\frac{\partial \Pi_1}{\partial p_{1t}} &= \Omega_1 - \left(\widetilde{\Pi_1} - p_{1t}\right)\frac{\partial \Omega_1}{\partial p_{1t}} - \Delta\Pi_1 \frac{\partial \Omega_0}{\partial p_{1t}} \\
\frac{\partial^2 \Pi_1}{\partial p_{1t}^2} &= 2\frac{\partial \Omega_1}{\partial p_{1t}} - \left[\widetilde{\Pi_1} - p_{1t}\right]\frac{\partial^2 \Omega_1}{\partial p_{1t}^2} - \Delta\Pi_1 \frac{\partial^2 \Omega_0}{\partial p_{1t}^2}. \\
\frac{\partial \Pi_2}{\partial p_{2t}} &= \Omega_2 - \left(\widetilde{\Pi_2} - p_{2t}\right)\frac{\partial \Omega_2}{\partial p_{2t}} - \Delta\Pi_2 \frac{\partial \Omega_0}{\partial p_{2t}} \\
\frac{\partial^2 \Pi_2}{\partial p_{2t}^2} &= 2\frac{\partial \Omega_2}{\partial p_{2t}} - \left[\widetilde{\Pi_2} - p_{2t}\right]\frac{\partial^2 \Omega_2}{\partial p_{2t}^2} - \Delta\Pi_2 \frac{\partial^2 \Omega_0}{\partial p_{2t}^2}.
\end{aligned}$$

For the ease of presentation, we omit all the function variables, and use $\Delta\Pi_1$ and $\Delta\Pi_2$ to denote $\Pi_1(x_1, x_2-1, t-1) - \Pi_1(x_1, x_2, t-1)$ and $\Pi_2(x_1-1, x_2, t-1) - \Pi_2(x_1, x_2, t-1)$ respectively. The same notations apply to the rest of the proof.

We first prove that Π_1 is unimodal in p_{1t}. Under the uniform valuation assumption, $\mathbf{H}_{DC}(\mathbf{v})$ can be rewritten as follows:

$$\mathbf{H}_{DC}(\mathbf{v}) = \begin{cases} (0,0) & \text{if } \max\{v_1-p_1, v_2-p_2\} < 0 \text{ and } \alpha_1 v_1 + \alpha_2 v_2 \le 2r; \\ (1,p_1) & \text{if } v_1 - p_1 \ge \max\{v_2-p_2, 0\} \text{ and } v_1 - v_2 \ge (p_1-2r)/\alpha_2; \\ (2,p_2) & \text{if } v_2 - p_2 \ge \max\{v_1-p_1, 0\} \text{ and } v_2 - v_1 \ge (p_2-2r)/\alpha_1; \\ (O,b^*) & \text{otherwise.} \end{cases} \qquad \text{(A10)}$$

To avoid trivial cases, assume that $2r_t \le 1$. Depending on the value of p_{1t}, its first order effect on Π_1 can take the following forms:

I. When $\alpha_1 p_{1t}+\alpha_2 p_{2t}<2r_t$ and $p_{1t}\le p_{2t}$, there are $\Omega_1 = 1-\dfrac{2p_{1t}+(1-p_{2t})^2}{2}$ and $\Omega_0=p_{1t}p_{2t}$. Hence,

$$\frac{\partial\Pi_1(\mathbf{x},t)}{\partial p_{1t}} = 1-2p_{1t}-\frac{(1-p_{2t})^2}{2}+\widetilde{\Pi_1}-\Delta\Pi_1 p_{2t} = -(p_{1t}-1-\Delta\Pi_1)^2+\frac{(1+\Delta\Pi_1)^2+1}{2}+\widetilde{\Pi_1}$$

II. When $\alpha_1 p_{1t}+\alpha_2 p_{2t}<2r_t$ and $p_{2t}\le p_{1t}$, there are $\Omega_1=\dfrac{(2p_{2t}+1-p_{1t})(1-p_{1t})}{2}$ and $\Omega_0 = p_{1t}p_{2t}$. Thus,

$$\frac{\partial\Pi_1(\mathbf{x},t)}{\partial p_{1t}} = \frac{3}{2}(1-p_{1t})^2+(2p_{2t}+\widetilde{\Pi_1}-1)(1-p_{1t})-p_{2t}(1+\Delta\Pi_1-\widetilde{\Pi_1})$$

III. When $\alpha_1 p_{1t}+\alpha_2 p_{2t}\ge 2r_t$ and $p_{1t}<2r_t$, there are $\Omega_1=-\frac{\alpha_1^2}{2\alpha_2^2}p_{1t}^2-p_{1t}(\frac{1}{\alpha_2}-2r_t\frac{\alpha_1}{\alpha_2^2})+\frac{1}{2}-\frac{2r_t^2}{\alpha_2^2}+\frac{2r_t}{\alpha_2}$ an $\Omega_0=-\dfrac{\alpha_1}{2\alpha_2}p_{1t}^2-\dfrac{\alpha_2}{2\alpha_1}p_{2t}^2+\dfrac{2r_t}{\alpha_2}p_{1t}+\dfrac{2r_t}{\alpha_1}p_{2t}-\dfrac{2r_t^2}{\alpha_1\alpha_2}$. Hence,

$$\begin{aligned}\frac{\partial\Pi_1(\mathbf{x},t)}{\partial p_{1t}} &= -\frac{3}{2}\frac{\alpha_1^2}{\alpha_2^2}p_{1t}^2-[\frac{2}{\alpha_2}-4\frac{r_t\alpha_1}{\alpha_2^2}-\frac{\alpha_1^2}{\alpha_2^2}\widetilde{\Pi_1}-\Delta\Pi_1\frac{\alpha_1}{\alpha_2}]p_{1t}\\ &+\frac{1}{2}+2\frac{r_t}{\alpha_2}-\frac{2r_t^2}{\alpha_2^2}+(\frac{1}{\alpha_2}-2\frac{r_t\alpha_1}{\alpha_2^2})\widetilde{\Pi_1}-2\frac{r_t}{\alpha_2}\Delta\Pi_1\end{aligned}$$

IV. When $\alpha_1 p_{1t}+\alpha_2 p_{2t}\ge 2r_t$ and $2r_t\le p_{1t}<2\frac{r_t}{\alpha_1}$, there are $\Omega_1 = \dfrac{(1+2\frac{\alpha_1}{\alpha_2})p_{1t}^2-(\frac{2}{\alpha_2}+4\frac{r_t}{\alpha_2})p_{1t}+1+4\frac{r_t}{\alpha_2}}{2}$ and $\Omega_0=-\dfrac{\alpha_1}{2\alpha_2}p_{1t}^2-\dfrac{\alpha_2}{2\alpha_1}p_{2t}^2+\dfrac{2r_t}{\alpha_2}p_{1t}+\dfrac{2r_t}{\alpha_1}p_{2t}-\dfrac{2r_t^2}{\alpha_1\alpha_2}$. Hence,

$$\begin{aligned}\frac{\partial\Pi_1(\mathbf{x},t)}{\partial p_{1t}} &= \frac{3}{2}(1+2\frac{\alpha_1}{\alpha_2})p_{1t}^2-[\frac{2}{\alpha_2}+4\frac{r_t}{\alpha_2}+(1+2\frac{\alpha_1}{\alpha_2})\widetilde{\Pi_1}-\Delta\Pi_1\frac{\alpha_1}{\alpha_2}]p_{1t}\\ &+\frac{1}{2}+2\frac{r_t}{\alpha_2}+(2\frac{r_t}{\alpha_2}+\frac{1}{\alpha_2})\widetilde{\Pi_1}-2\frac{r_t}{\alpha_2}\Delta\Pi_1\end{aligned}$$

V. When $2\frac{r_t}{\alpha_1}\le p_{1t}<\alpha_2+2r_t$, there are $\Omega_1=\dfrac{(1-\frac{p_{1t}-2r_t}{\alpha_2})^2}{2}$ and Ω_0 is independent of p_{1t}. Thus,

$$\frac{\partial\Pi_1(\mathbf{x},t)}{\partial p_{1t}} = \frac{(1-\frac{p_{1t}-2r_t}{\alpha_2})^2}{2}+\frac{1}{\alpha_2}(\widetilde{\Pi_1}-p_{1t})(1-\frac{p_{1t}-2r_t}{\alpha_2})$$

VI. When $\alpha_2+2r_t\le p_{1t}\le 1$, there are $\Omega_1=0$ and Ω_0 independent of p_{1t}. Thus, $\dfrac{\partial\Pi_1(\mathbf{x},t)}{\partial p_{1t}}=0$.

We next prove the unimodal by showing that there exists an $p_{1t}^*\in[0,1]$ such that $\dfrac{\partial\Pi_1(\mathbf{x},t)}{\partial p_{1t}}\ge 0$ if $0\le p_{1t}\le p_{1t}^*$, and $\dfrac{\partial\Pi_1(\mathbf{x},t)}{\partial p_{1t}}\le 0$ if $p_{1t}^*\le p_{1t}\le 1$.

First note that $\dfrac{\partial\Pi_1(\mathbf{x},t)}{\partial p_{1t}}$ is continuous in Scenario I and II. In Scenario I, it increases in p_{1t} and

$$\frac{\partial\Pi_1(\mathbf{x},t)}{\partial p_{1t}}|_{p_{1t}=0} = \frac{1}{2}-\frac{p_{2t}^2}{2}+(1-\Delta\Pi_1)p_{2t}+\widetilde{\Pi_1}>0.$$

So $\dfrac{\partial \Pi_1(\mathbf{x},t)}{\partial p_{1t}} > 0$ in this scenario.

In Scenario II, $\dfrac{\partial \Pi_1(\mathbf{x},t)}{\partial p_{1t}}$ is convex. Since $-p_{2t}(1+\Delta\Pi_1-\widetilde{\Pi_1})<0$, at least one (if any) of the roots of the first-order condition (FOC) should be greater than 1. Also, as

$$\frac{\partial \Pi_1(\mathbf{x},t)}{\partial p_{1t}}|_{p_{1t}=0} = \frac{1}{2}+4r_t+(4r_t+2)\widetilde{\Pi_1}-4r_t\Delta\Pi_1>0,$$

there is one $p_{1t}\in[0,1]$ at which the FOC is achieved, denoted as p_{1t}^{II}.

In Scenario III, $\dfrac{\partial \Pi_1(\mathbf{x},t)}{\partial p_{1t}}$ is concave and

$$\frac{\partial \Pi_1(\mathbf{x},t)}{\partial p_{1t}}|_{p_{1t}=0} = \frac{1}{2}+2\frac{r_t}{\alpha_2}-\frac{2r_t^2}{\alpha_2^2}+(-2\frac{r_t\alpha_1}{\alpha_2}+1+\frac{\alpha_1}{\alpha_2})\widetilde{\Pi_1}-2\frac{r_t}{\alpha_2}\Delta\Pi_1>0$$

Then exactly one root of FOC is positive, which is denote as p_{1t}^{III}.

In Scenario IV, $\dfrac{\partial \Pi_1(\mathbf{x},t)}{\partial p_{1t}}$ is convex and

$$\begin{aligned}\frac{\partial \Pi_1(\mathbf{x},t)}{\partial p_{1t}}|_{p_{1t}=0} &= \frac{1}{2}+4r_t+(4r_t+2)\widetilde{\Pi_1}-4r_t\Delta\Pi_1>0\\ \frac{\partial \Pi_1(\mathbf{x},t)}{\partial p_{1t}}|_{p_{1t}=1} &= (1-4r_t)(1-\widetilde{\Pi_1}-\Delta\Pi_1)<0\end{aligned}$$

Thus, there is one $p_{1t}\in[0,1]$ at which FOC is achieved, denoted as p_{1t}^{IV}.

In scenario V, the FOC can be achieved at $p_{1t}=\dfrac{\alpha_2+2r_t+2\widetilde{\Pi_1}}{3}$ and $p_{1t}=\alpha_2+2r_t$, where $\dfrac{\alpha_2+2r_t+2\widetilde{\Pi_1}}{3}\le\alpha_2+2r_t$. Denote $p_{1t}^V=\dfrac{\alpha_2+2r_t+2\widetilde{\Pi_1}}{3}$. If $p_{1t}^V\le\frac{2r_t}{\alpha_1}$, then $\dfrac{\partial \Pi_1(\mathbf{x},t)}{\partial p_{1t}}$ is constantly negative in Scenario V. Otherwise, it can be shown that $\dfrac{\partial \Pi_1(\mathbf{x},t)}{\partial p_{1t}}$ in Scenario IV will be 0 at $p_{1t}=\frac{2r_t}{\alpha_1}$, i.e., $p_{1t}^{IV}=\frac{2r_t}{\alpha_1}$.

Thus, p_{1t}^* can only be among p_{1t}^{II}, p_{1t}^{III}, p_{1t}^{IV} and p_{1t}^V. It can be shown that p_{1t}^{III} and p_{1t}^{IV} cannot both be feasible at the same time, i.e., $p_{1t}^{II}\le 2r_t\le p_{1t}^{IV}$ cannot hold true. Otherwise, it can be verified that

$$\frac{\partial \Pi_1(\mathbf{x},t)}{\partial p_{1t}}^{III}|_{p_{1t}=2r_t}-\frac{\partial \Pi_1(\mathbf{x},t)}{\partial p_{1t}}^{IV}|_{p_{1t}=2r_t}\sim 1+\alpha_1^2-\alpha_2^2\ge 0$$

However, for what we have discussed about $\dfrac{\partial \Pi_1(\mathbf{x},t)}{\partial p_{1t}}$, there should be $\dfrac{\partial \Pi_1(\mathbf{x},t)}{\partial p_{1t}}^{IV}|_{p_{1t}=2r_t}\ge \dfrac{\partial \Pi_1(\mathbf{x},t)}{\partial p_{1t}}^{IV}|_{p_{1t}=p_{1t}^{IV}}=0=\dfrac{\partial \Pi_1(\mathbf{x},t)}{\partial p_{1t}}^{III}|_{p_{1t}=p_{1t}^{III}}\ge\dfrac{\partial \Pi_1(\mathbf{x},t)}{\partial p_{1t}}^{III}|_{p_{1t}=2r_t}$. The same reasoning applies to p_{1t}^{II} vs. p_{1t}^{III} and p_{1t}^{IV} vs. p_{1t}^V .

Therefore, it can be concluded that at most one of p_{1t}^{II}, p_{1t}^{III}, p_{1t}^{IV} and p_{1t}^V will be feasible and $\dfrac{\partial \Pi_1(\mathbf{x},t)}{\partial p_{1t}}\ge 0$ (resp. ≤ 0) when p_{1t} is less (resp. greater) than it. If none of the p_{1t}^{II}, p_{1t}^{III}, p_{1t}^{IV} or p_{1t}^V is feasible, the profit function is monotone on $[0,1]$ and $\Pi_1(\mathbf{x},t)$ is still unimodal in p_{1t}.

The proof for $\Pi_2(\mathbf{x},t)$ can be done in a similar fashion. Therefore, there exists a pure NE for the posted prices (p_{1t},p_{2t}). □

Proof of Corollary 2:

(i) At $t=1$, the marginal value of inventory $\widetilde{\Pi}_i(\mathbf{x}, t-1)$ is zero for either seller. By Proposition 2, the reservation price is $r^* = \max_i \widetilde{\Pi}_i(\mathbf{x}, t-1) = 0$.

(ii) We can show that, at any time t, if $x_1 \geq t$ and $x_2 \geq t$,

$$\Pi_1(x_1, x_2, t) = \Pi_2(x_1, x_2, t) = \pi_t \tag{A11}$$

for some $\pi_t > 0$. Hence $\widetilde{\Pi}_i(\mathbf{x}, t) = 0$ for any $x_1 \geq t$ and $x_2 \geq t$, which immediately leads to $r^* = 0$.

We show (A11) by induction. First, it is apparent that (A11) holds for $t=1$. Now, suppose (A11) holds for all $t < T$. We then have $\Pi_1(x_1-1, x_2, T-1) = \Pi_1(x_1-1, x_2-1, T-1) = \Pi_2(x_1, x_2-1, T-1) = \Pi_2(x_1-1, x_2-1, T-1)$ for any $x_1 \geq T$ and $x_2 \geq T$. Thus, $\widetilde{\Pi}_i(x_1, x_2, T-1) = 0$ for $i = 1, 2$ and

$$r^*(x_1, x_2, T) = 0, \qquad \forall x_1 \geq T, x_2 \geq T. \tag{A12}$$

(A12) implies that, when both sellers oversupply, the price of the opaque goods will remain zero until one seller's inventory becomes lower than the potential demand. The expected profit for seller 1 is then

$$\begin{aligned}\Pi_1(\mathbf{x}, T) &= [\,p_{1T} + \Pi_1(x_1-1, x_2, T-1)\,]\,\Omega_1(\mathbf{r}_t, \mathbf{p}_t) + \Pi_1(x_1-1, x_2, T-1)\Omega_O(\mathbf{r}_t, \mathbf{p}_t) \\ &\quad + \Pi_1(x_1, x_2-1, T-1)\Omega_2(\mathbf{r}_t, \mathbf{p}_t) + \Pi_1(x_1, x_2, T-1)\Omega_0(\mathbf{r}_t, \mathbf{p}_t) \\ &= p_{1T}\Omega_1(\mathbf{r}_t, \mathbf{p}_t) + \pi_{T-1}, \end{aligned} \tag{A13}$$

which does not depend on $\mathbf{x}$ for $x_1 \geq T$ and $x_2 \geq T$. Similarly, $\Pi_2(\mathbf{x}, T) = p_{2T}\Omega_2(\mathbf{r}_t, \mathbf{p}_t) + \pi_{T-1}$. When $r = 0$, $\Omega_i(\mathbf{r}_t, \mathbf{p}_t) = \dfrac{(1 - p_{it}/\alpha_i)^2}{2}$ for $i = 1, 2$. It is then straightforward that in equilibrium there should be $p_{1T} = p_{2T}$ and $\Pi_1(\mathbf{x}, T) = \Pi_2(\mathbf{x}, T)$. Thus, (A11) also holds for $t = T$. This completes the proof. $\square$

Proof of Proposition 3: Apparently there is oversupply when $x_1 = x_2 = \infty > T$. For a similar analysis as in the proof of Corollary 2 (ii), under DC the reservation price is $r_i^* = 0$ and expected profit $\Pi_i^{DC}(x_1, x_2, t) = \pi_t^{DC}$; under SC the expected profit $\Pi_i^{SC}(x_1, x_2, t) = \pi_t^{SC}$, for $i = 1, 2$ and any t;

Specifically for $T = 1$, the optimal posted price under DC is set by $p_{iT}^{DC} = \arg\max_{0 \leq p_{iT} \leq 1} p_{iT} \frac{(1 - p_{iT}/\alpha_i)^2}{2} = \frac{\alpha_i}{3}$ and $\Pi_{iT}^{DC} = \frac{2\alpha_i}{27}$. However, under SC the equilibrium posted price is $p_{iT}^{SC} = \sqrt{2} - 1$ and $\Pi_{iT}^{SC} = 3 - 2\sqrt{2} > 2/27 > \Pi_{iT}^{DC} = 2\alpha_i/27$. Therefore at $T = 1$, the equilibrium channel structure is SC.

Suppose SC is the equilibrium channel structure for all $T < t$. At $T = t$ under DC, by (A13) the expected profit for seller i is $\Pi_{it}^{DC} = 2\alpha_i/27 + \pi_{t-1}^{DC}$. At $T = t$ under SC, the expected profit for seller i is $\Pi_{it}^{SC} = 3 - 2\sqrt{2} + \pi_{t-1}^{SC}$. Since SC is the equilibrium at $t-1$, there should be $\pi_{t-1}^{SC} \geq \pi_{t-1}^{DC}$. Thus $\Pi_{it}^{SC} > \Pi_{it}^{DC}$. Hence SC is the equilibrium at $T = t$.

This proves the claim that SC is the equilibrium when supply is unlimited.

□

Proof of Proposition 4: To prove that SDCi is an equilibrium when $x_1 \to \infty$ and $x_2 \to \infty$, it is suffice to show that $\Pi_1^{10}(\infty,\infty,T) \geq \Pi_1^{00}(\infty,\infty,T)$ and $\Pi_1^{01}(\infty,\infty,T) \geq \Pi_1^{11}(\infty,\infty,T)$ for any $T > 0$.

At $T = 1$, it can be verified that reservation price will be set high under SDCi structure and all transactions will be realized via the direction channel. By the proof of Proposition 3 there are $\Pi_1^{10}(\infty,\infty,1) = \Pi_1^{00}(\infty,\infty,1) = 3 - 2\sqrt{2}$ and $\Pi_1^{01}(\infty,\infty,1) = 3 - 2\sqrt{2} \geq \Pi_1^{11}(\infty,\infty,1) = 2\alpha_i/27$. Thus SDC$i$ is an NE when $T = 1$.

Suppose the statements hold true for $T < t$. Then at $T = t$, by Corollary 2 (ii) the reservation price under DC is 0. Therefore.

$$\begin{aligned}
\Pi_1^{11}(\infty,\infty,t) = \quad & \frac{1}{2}\Omega_O(\mathbf{r}_t,\mathbf{p}_t)\Pi_1^{11}(\infty,\infty,t-1) + \Omega_1(\mathbf{r}_t,\mathbf{p}_t)\left[p_{1t} + \Pi_1^{11}(\infty,\infty,t-1)\right] \\
& + [\Omega_2(\mathbf{r}_t,\mathbf{p}_t) + \frac{1}{2}\Omega_O(\mathbf{r}_t,\mathbf{p}_t)]\Pi_1^{11}(\infty,\infty,t-1) + \Omega_0(\mathbf{r}_t,\mathbf{p}_t)\Pi_1^{11}(\infty,\infty,t-1) \\
= \; & \Pi_1^{11}(\infty,\infty,t-1) + \Omega_1(\mathbf{0},\mathbf{p}_t)p_{1t}
\end{aligned}$$

Thus in the equilibrium of DC, there will be $\Pi_1^{11}(\infty,\infty,t) = \Pi_1^{11}(\infty,\infty,t-1) + \Pi_1^{11}(\infty,\infty,1)$.

Similarly for SDC2,

$$\begin{aligned}
\Pi_1^{01}(\infty,\infty,t) = \quad & \Omega_1(\mathbf{r}_t,\mathbf{p}_t)\left[p_{1t} + \Pi_1^{01}(\infty,\infty,t-1)\right] + [\Omega_2(\mathbf{r}_t,\mathbf{p}_t) + \Omega_O(\mathbf{r}_t,\mathbf{p}_t)]\Pi_1^{01}(\infty,\infty,t-1) \\
& + \Omega_0(\mathbf{r}_t,\mathbf{p}_t)\Pi_1^{01}(\infty,\infty,t-1) \\
= \; & \Pi_1^{01}(\infty,\infty,t-1) + \Omega_1(\mathbf{r}_t,\mathbf{p}_t)p_{1t}
\end{aligned}$$

and at equilibrium there is $\Pi_1^{01}(\infty,\infty,t) = \Pi_1^{01}(\infty,\infty,t-1) + \Pi_1^{01}(\infty,\infty,1)$.

Since $\Pi_1^{01}(\infty,\infty,T) \geq \Pi_1^{11}(\infty,\infty,T)$ for any $T < t$, there should be $\Pi_1^{01}(\infty,\infty,t) = \Pi_1^{01}(\infty,\infty,t-1) + \Pi_1^{01}(\infty,\infty,1) \geq \Pi_1^{11}(\infty,\infty,t-1) + \Pi_1^{11}(\infty,\infty,1) = \Pi_1^{11}(\infty,\infty,t)$. Thus $\Pi_1^{01}(\infty,\infty,T) \geq \Pi_1^{11}(\infty,\infty,T)$ also holds for $T = t$. The same argument applies to $\Pi_1^{10}(\infty,\infty,t) \geq \Pi_1^{00}(\infty,\infty,t)$. Therefore, the statements also hold for $T = t$. This proves that in general SDCi is an equilibrium when inventory is unlimited. □

Proof of Proposition 5:

(i) For vertical differentiation, consider that $\{\mathbf{v}_t : v_{it} - v_{jt} = \bar{v}_t\}$ for some constant $\bar{v}_t > 0$. By $\mathbf{H}_{DC}(\mathbf{v})$ in §4.3, a customer will consistently buy from the *same* type of channel (i.e., from seller i, or seller j, or the NYOP channel) or leave empty handed. (*)

• Consider $t = 1$, the last period sales. Due to Theorem 1, the statement apparently holds when one seller is out of stock. Now, suppose both sellers are in stock. Corollary 2 implies that both sellers has marginal inventory value zero thus $r_1 = 0$ hence no seller earns from the NYOP channel. By (*), sellers' should set posted prices to induce the customer purchase through direct channel ultimately.

We first argue that a customer will not buy directly from seller j. First, there is always a strategy in which seller i can set his posted prices as $p_{i1} = p_{j1} + v_{i1} - v_{j1} - \varepsilon = p_{j1} + \bar{v}_t - \varepsilon$ for some $\varepsilon > 0$ such that the last-period customer will always prefer to buy from seller i at p_{i1}. Second, it is optimal for seller i to set such a posted price in winning the potential customer, since direct channel is the only place that generates income. The competition then drives the period-1 posted prices for i and j to $p_{i1} = \bar{v}_1 - \varepsilon$ and $p_{j1} = 0$ respectively.

Next, to ensure that the customer will prefer buying directly from seller i than obtaining opaque product from NYOP channel, the posted price for i should also satisfy $\alpha_i v_{i1} + \alpha_j v_{j1} \leq v_{i1} - p_{i1}$, hence $p_{i1} \leq \alpha_j \bar{v}_1$. The equilibrium at $t = 1$ is therefore $p_{i1} = \alpha_j \bar{v}_1$ and $p_{j1} = 0$. The expected profit for seller i is $\alpha_j \bar{v}_1$. seller j and the NYOP firm earns zero.

The statement is true for $t = 1$.

• Now consider $t > 1$. Denote $v^r_{it} = r^*_t + \dfrac{G(r^*_t)}{g(r^*_t)} + \alpha_j \bar{v}_t$ and $v^r_{jt} = r^*_t + \dfrac{G(r^*_t)}{g(r^*_t)} - \alpha_i \bar{v}_t$. There is $v^r_{it} - v^r_{jt} = \bar{v}_t$. Then, if $v_{it} < p_{it}$ and $v_{jt} < p_{jt}$, customers with $v_{it} \geq v^r_{it}$ or $v_{jt} \geq v^r_{jt}$ will bid above the reservation price r^*_t and purchase through the NYOP channel. Other customers cannot afford either direct channel and will leave empty handed.

If $v_{it} \geq p_{it}$ and $\bar{v}_t \geq p_{it} - p_{jt}$, analysis in §4.3 suggests that the optimal bid satisfies $\min\{v_{it}, p_{it}\} = b^* + \dfrac{G(b^*)}{g(b^*)} + \alpha_j \bar{v}_t$. Thus those with $\min\{v_{it}, p_{it}\} < v^r_{it}$ will bid below r^*_t and will purchase through direct channel i in the end.

If $v_{jt} \geq p_{jt}$ and $\bar{v}_t \leq p_{it} - p_{jt}$, the optimal bid satisfies $\min\{v_{jt}, p_{jt}\} = b^* + \dfrac{G(b^*)}{g(b^*)} - \alpha_i \bar{v}_t$, and those with $\min\{v_{jt}, p_{jt}\} < v^r_{jt}$ will bid below r^*_t and will purchase through direct channel j in the end.

With a slight abuse of the notation, denote $\widetilde{\Pi}_{it} = \widetilde{\Pi}_i(\mathbf{x}, t)$ and $\widetilde{\Pi}_{jt} = \widetilde{\Pi}_j(\mathbf{x}, t)$. We next characterize the equilibrium pricing strategy when seller i is the opaque seller (i.e., $\widetilde{\Pi}_{it} \leq \widetilde{\Pi}_{jt}$). For seller i's optimal pricing response given p_{jt}:

• First consider the case when $p_{jt} \geq v^r_{jt}$. If $p_{it} \geq v^r_{it}$, customers with $v_i \geq v^r_{it}$ will purchase through the NYOP channel, and those with $v_i < v^r_{it}$ are not accepted by the NYOP channel and cannot afford either direct channel, hence will left empty-handed. Seller i's expected profit is $\Pi_1(\mathbf{x}, t) =$

$\Pi_1(x_1-1,x_2,t-1)\dfrac{1-v_{it}^r}{1-\bar{v}_t}+\Pi_1(x_1,x_2,t-1)\dfrac{v_{it}^r-\bar{v}_t}{1-\bar{v}_t}$. If $p_{it}<v_{it}^r$, bids from all customers will be rejected. The only realized sales are through direct channel i. The expected profit for seller i is $\Pi_1(\mathbf{x},t)=[\,p_{it}+\Pi_1(x_1-1,x_2,t-1)\,]\,\dfrac{1-p_{it}}{1-\bar{v}_t}+\Pi_1(x_1,x_2,t-1)\dfrac{p_{it}-\bar{v}_t}{1-\bar{v}_t}$. Apparently, it is optimal for seller i to set $p_{it}=v_{it}^r-\epsilon$ for some small $\epsilon>0$ in this scenario.

- For the case where $p_{jt}<v_{jt}^r$, if $p_{it}>p_{jt}+\bar{v}_t$, then all bids will be rejected and customers with $v_{jt}\geq p_{jt}$ will buy through direct channel j in the end. Seller i's expected profit is $\Pi_1(\mathbf{x},t)=\Pi_1(x_1,x_2-1,t-1)\dfrac{1-p_{jt}-\bar{v}_t}{1-\bar{v}_t}+\Pi_1(x_1,x_2,t-1)\dfrac{p_{jt}}{1-\bar{v}_t}$. If $p_{it}<p_{jt}+\bar{v}_t$, all bids will be rejected and customers with $v_{it}\geq p_{it}$ will buy through direct channel i in the end. Seller i's expected profit is $\Pi_1(\mathbf{x},t)=[\,p_{it}+\Pi_1(x_1-1,x_2,t-1)\,]\,\dfrac{1-p_{it}}{1-\bar{v}_t}+\Pi_1(x_1,x_2,t-1)\dfrac{p_{it}-\bar{v}_t}{1-\bar{v}_t}$. Comparing the two options, it is optimal for seller i to set $p_{it}=p_{jt}+\bar{v}_t-\epsilon$ for some small $\epsilon>0$ if $p_{jt}+\bar{v}_t>\widetilde{\Pi}_{it}$.

Overall, $p_{it}^*(p_{jt})=\min\{\max\{\widetilde{\Pi}_{it},p_{jt}+\bar{v}_t\},v_{it}^r\}$. Similarly for seller j, it can be verified that $p_{jt}^*(p_{it})=\min\{\max\{\widetilde{\Pi}_{jt},p_{it}-\bar{v}_t\},v_{jt}^r\}$. The same pair of response functions hold when seller j is the opaque product provider (i.e., $\widetilde{\Pi}_{it}\geq\widetilde{\Pi}_{jt}$). In either case, sales are realized through one direct channel only, and no sales will be through the NYOP channel.

(ii) For the horizontal differentiation case, consider that $\{\alpha_i v_i+\alpha_j v_j=\bar{v}_t\}$ for some $\bar{v}_t>0$. $\mathbf{H}_{DC}(\mathbf{v})$ suggests that there are *multiple* types of channel a customer can possibly end up with, depending on her valuation realization (v_{it},v_{jt}).

Denote $v_t^*=r_t^*+\dfrac{G(r_t^*)}{g(r_t^*)}$, where $r_t^*=\min\{\widetilde{\Pi}_{it},\widetilde{\Pi}_{jt}\}$ is the reservation price in period t. Then, if $\bar{v}_t\geq v_t^*$, all customers will bid above r_t^* when $v_{it}-p_{it}<0$ and $v_{jt}-p_{jt}<0$. We next show that if $\bar{v}_t\geq v_t^*$, then there exists some conditions under which $\Omega_O>0$ with equilibrium pricing (p_{it}^*,p_{jt}^*).

We start with analyzing seller i's optimal pricing response given p_{jt}:

- If $p_{it}\leq v_t^r-p_{jt}$, then $\Omega_i=\alpha_j\dfrac{v_t^r-\alpha_i p_{it}+\alpha_i p_{jt}}{v_t^r}$ and $\Omega_0=\Omega_O=0$. As analyzed in the proof of Theorem **??**,

$$\frac{\partial\Pi_i}{\partial p_{it}}=\Omega_i-\left(\widetilde{\Pi_i}-p_{it}\right)\frac{\partial\Omega_i}{\partial p_{it}}-\Delta\Pi_i\frac{\partial\Omega_0}{\partial p_{it}}\tag{A14}$$

where $\Delta\Pi_i=\Pi_i(x_i,x_j-1,t-1)-\Pi_i(x_i,x_j,t-1)$. Then,

$$\begin{aligned}\frac{\partial\Pi_i}{\partial p_{it}}&=\alpha_j\frac{v_t^r-\alpha_i p_{it}+\alpha_i p_{jt}}{v_t^r}-\left(p_{it}-\widetilde{\Pi_i}\right)\frac{\alpha_i\alpha_j}{v_t^r}=\frac{\alpha_i\alpha_j}{v_t^r}\left(\frac{v_t^r}{\alpha_i}-2p_{it}+p_{jt}+\widetilde{\Pi}_{it}\right)\\&\geq\frac{\alpha_i\alpha_j}{v_t^r}\left(\frac{v_t^r}{\alpha_i}-2v_t^r+3p_{jt}+\widetilde{\Pi}_{it}\right)\end{aligned}$$

- If $p_{it}>v_t^r-p_{jt}$, then $\Omega_i=\dfrac{v_t^r-\alpha_i p_{it}}{v_t^r}$, $\Omega_j=\dfrac{v_t^r-\alpha_j p_{jt}}{v_t^r}$, $\Omega_O=\dfrac{\alpha_i p_{it}+\alpha_j p_{jt}-v_t^r}{v_t^r}$ and $\Omega_0=0$. Thus,

$$\frac{\partial\Pi_i}{\partial p_{it}}=\frac{v_t^r-\alpha_i p_{it}}{v_t^r}-\left(p_{it}-\widetilde{\Pi_i}\right)\frac{\alpha_i}{v_t^r}=\frac{\alpha_i}{v_t^r}\left(\frac{v_t^r}{\alpha_i}-2p_{it}+\widetilde{\Pi}_{it}\right)$$

The first order derivative is greater than zero if $p_{it} < \dfrac{\frac{v_t^r}{\alpha_i} + \widetilde{\Pi}_{it}}{2}$.

The same analysis can be done for seller j. It can be verified that the equilibrium is

$$(p_{it}^*, p_{jt}^*) = \left(\frac{\frac{v_t^r}{\alpha_i} + \widetilde{\Pi}_{it}}{2}, \frac{\frac{v_t^r}{\alpha_j} + \widetilde{\Pi}_{jt}}{2} \right),$$

at which $\Omega_O = \dfrac{\alpha_i \widetilde{\Pi}_{it} + \alpha_j \widetilde{\Pi}_{jt}}{2v_t^r} > 0$ if one of $\widetilde{\Pi}_{it}$ and $\widetilde{\Pi}_{jt}$ is nonzero. Thus the NYOP channel earns non-zero profit when $v_t^r \leq \bar{v}_t$ and $\max\{\widetilde{\Pi}_{it}, \widetilde{\Pi}_{jt}\} > 0$.

□